1. Find the energy stored in the region $r > a$ of a point charge Q placed at the origin. [10]

Solution.

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}v = \frac{1}{2}\epsilon_0\int E^2\,\mathrm{d}v$$

$$\mathbf{E} = \frac{Q\mathbf{R}}{4\pi\epsilon_0 R^3}$$

But $\mathrm{d}v = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ and,

$$\begin{aligned}\mathbf{R} &= \mathbf{r}-\mathbf{r}_0\\ &= (x,y,z)\\ &= (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\\ R^3 &= r^3.\end{aligned}$$

Therefore

$$\mathbf{E} = \frac{Q(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)}{4\pi\epsilon_0 r^3}$$

$$\begin{aligned}E^2 &= \frac{Q^2}{(4\pi\epsilon_0)^2 r^4}(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta)\\ &= \frac{Q^2}{(4\pi\epsilon_0)^2 r^4}(\sin^2\theta+\cos^2\theta)\\ &= \frac{Q^2}{(4\pi\epsilon_0 r^2)^2}\end{aligned}$$

$$\begin{aligned}W &= \frac{Q^2}{2}\frac{\epsilon_0}{(4\pi\epsilon_0)^2}\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\frac{1}{r^2}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{Q^2}{32\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\sin\theta\left(-\frac{1}{r}\Big|_a^{\infty}\right.\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{Q^2}{32\pi^2\epsilon_0 a}\int_0^{2\pi}(-\cos\theta|_0^{\pi}\,\mathrm{d}\phi\\ &= \frac{(1+1)}{32\pi^2\epsilon_0 a}(\phi|_0^{2\pi}\\ &= \frac{4\pi Q^2}{32\pi^2\epsilon_0 a}\\ &= \frac{Q^2}{8\pi\epsilon_0 a}\end{aligned}$$

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Alternatively, $r > a$;

$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v\mathbf{a}_r$$

$$\Psi = \oint \mathbf{D}\cdot\mathrm{d}\mathbf{S} = D_r 4\pi r^2$$

$$\begin{aligned}D_r 4\pi r^2 &= Q\\ D_r &= \frac{Q}{4\pi r^2}\\ \mathbf{D} &= \frac{Q}{4\pi r^2}\mathbf{a}_r\end{aligned}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

$$E^2 = \frac{Q^2}{16\pi^2\epsilon_0^2 r^4}$$

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\begin{aligned}
W &= \frac{1}{2}\epsilon_0 \int \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{Q^2}{32\pi^2\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_a^{\infty} \frac{1}{r^2} \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{Q^2}{8\pi\epsilon_0 a}
\end{aligned}
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